Higher order corrections to primordial spectra from cosmological inflation

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Abstract

We calculate power spectra of cosmological perturbations at high accuracy for two classes of inflation models. We classify the models according to the behaviour of the Hubble distance during inflation. Our approximation works if the Hubble distance can be approximated either to be a constant or to grow linearly with cosmic time. Many popular inflationary models can be described in this way, e.g., chaotic inflation with a monomial potential, power-law inflation and inflation at a maximum. Our scheme of approximation does not rely on a slow-roll expansion. Thus we can make accurate predictions for some of the models with large slow-roll parameters.

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1 Introduction

The most important prediction of cosmological inflation [1] (besides the spatial flatness of the universe) is the generation of primordial fluctuations of matter and space-time [2]. These fluctuations seed the formation of large scale structure and give rise to anisotropies in the cosmic microwave background (CMB). Recent CMB observations [3] have clearly detected the first three acoustic peaks. In order to explain these observations a dominant contribution of adiabatic perturbations is necessary, as predicted by cosmological inflation. The position of these peaks is consistent with a spatially flat universe. The level of accuracy of current experiments is of the order of 10%, which roughly corresponds to 10% uncertainty for the best determined cosmological parameters. Future experiments [4] will increase this accuracy to the limit of the cosmic variance (of the order of 1% at the arcminute scale).

For a reliable comparison of inflationary predictions and CMB data we need analytical predictions that can meet the accuracy of the observations. The primordial fluctuations are characterised in terms of power spectra of scalar and tensor perturbations (inflation driven by a scalar field predicts that there are no vector perturbations). Only a limited number of very special inflationary models is known for which the power spectra can be calculated exactly [5]. Thus we have to use some approximation or we have to rely on numerical calculations [6]. The state of the art of the analytical calculation of the spectral amplitudes are the approximate expressions due to Stewart and Lyth [7], which are first order expressions in terms of the so-called, slow-roll parameters. These expressions have been tested in Refs. [8, 9] to be precise enough to match the accuracy of current observations, given the conditions for slow roll are met. However, it has been pointed out that for the analysis of data from future observations, more precise analytical expressions are compulsory [9]. Moreover, there are inflationary models that do not belong to the class of slow-roll models. The approach by Stewart and Lyth [7] includes several approximations. In a first step the mode equations of the fluctuations are approximated by Bessel differential equations. With this aim, one has to assume that the slow-roll parameters are constant, which is a consistent procedure up to first order in the slow-roll parameters only. A second approximation is made by assuming that the dominant modes take their superhorizon values at the moment of horizon crossing, which is arbitrarily fixed as $k = aH$. Martin and Schwarz [9] closed a gap in the derivation of this approximation, showing that the time of matching the Bessel function

solution to the superhorizon solution has to coincide with the time when the Hubble rate and the slow-roll parameters are evaluated. Wang, Mukhanov and Steinhardt [10] have shown that the method of Stewart and Lyth [7] does not allow to improve the Bessel function approximation to higher orders. Recently, Stewart and Gong [11] introduced a method to calculate the spectra to higher orders in the slow-roll parameters for slow-roll inflation. This new method does no longer refer to the Bessel function approximation, but instead uses an approximation based on Greens functions. They calculate the scalar spectral amplitude at second order in the slow-roll parameters.

In this paper we present an approximation that does not rely on the slow-roll approximation and has the advantage that models for which some of the slow-roll parameters take large values can be treated. We calculate the amplitudes with high precision for two families of inflationary models, models that are characterised by an almost constant Hubble distance during inflation and models that are characterised by an almost linearly growing Hubble distance. The common link between both families is that their spectral indices are almost constant, even for large values of some of the slow-roll parameters. Our approximation makes use of the Bessel function approximation, but avoids the argument by Wang, Mukhanov and Steinhardt [10] by putting constraints on the relative magnitude of the various slow-roll parameters.

The traditional set of slow-roll parameters (see e.g. [12]) is not well suited for the new approximation. Although we hesitated to introduce yet another set of parameters that control the dynamics of inflation, our new parameters have two major advantages. Firstly, their definition is simpler than other definitions of slow-roll parameters, and, secondly, the new definition allows a very transparent physical interpretation of the parameters: the parameters control how the Hubble distance behaves during inflation and thus we call them the horizon flow functions (or parameters when we evaluate these functions at a certain moment of time). We have presented a variant of our new approach using the traditional notation in Ref. [13].

This paper is organised as follows: In section 2 we introduce the horizon flow functions and establish the link to the slow-roll parameters. The core of the paper is section 3, where we calculate the scalar power spectrum in the constant-horizon and in the growing-horizon approximations up to third order. The tensor power spectra are calculated in section 4. In section 5 we present the corresponding ‘consistency relations of inflation’. We finally compare our results to the result of Stewart and Gong [11] and show that both results are consistent, which is a nontrivial check of both calculations.

2 Horizon flow

For the reasons explained in the introduction we define a set of horizon flow functions starting from

$$\epsilon_0 \equiv \frac{d_{\rm H}(N)}{d_{\rm Hi}}, \tag{1}$$

where $d_{\rm H} \equiv 1/H$ denotes the Hubble distance, $N \equiv \ln(a/a_{\rm i})$ the number of e-folds since some initial time $t_{\rm i}$, and $d_{\rm Hi} \equiv d_{\rm H}(t_{\rm i})$. (Note that usually the number of e-folds is counted backward in time, we count it forward, i.e., $N(t_{\rm i}) = 0$.) The quantity $d_{\rm H}$ is commonly called "horizon" because it is a good estimate of the size of the region that may be in causal contact within one expansion time.

Further we define a hierarchy of functions in a systematic way by

$$\epsilon_{m+1} \equiv \frac{{\rm d} \ln |\epsilon_m|}{{\rm d}N}, \qquad m \geq 0. \tag{2}$$

According to this definition,

$$\epsilon_1 \equiv \frac{{\rm d} \ln d_{\rm H}}{{\rm d}N}, \tag{3}$$

measures the logarithmic change of the Hubble distance per e-fold of expansion. Inflation happens for $\epsilon_1 < 1$ (equivalent to $\ddot{a} > 0$) and $\epsilon_1 > 0$ from the weak energy condition (for a spatially flat universe). For $m > 1$, ϵ_m may take any real value. Expressions (2) define a flow in the space $\{\epsilon_m\}$ with the cosmic time being the evolution parameter. This flow is described by the equations of motion

$$\epsilon_0 \dot{\epsilon_m} - \frac{1}{d_{\rm Hi}} \epsilon_m \epsilon_{m+1} = 0. \tag{4}$$

For $m = 0(1)$ we find $\epsilon_1 = \dot{d_{\rm H}}$ and $\epsilon_1\epsilon_2 = d_{\rm H}\ddot{d_{\rm H}}$ respectively, which describe the time evolution of the horizon.

Let us stress the advantages of definition (2) over other definitions of the slow-roll parameters [7, 9, 12]. First of all, the physical interpretation of the functions given by Eq. (2) and equations of motion (4) is straightforward and model independent (not restricted to models with a single scalar field). Secondly, the notation is concise, leading to significant simplification in the involved expressions. Thirdly, the definition is easy to memorise. The link between the first three horizon flow functions and various definitions of the corresponding slow-roll parameters is presented in table 1.

this work	Lidsey et al. [12]	Martin & Schwarz [9]	Stewart & Lyth [7][a] Stewart & Gong [11]
ϵ_1	ϵ	ϵ	ϵ_1
ϵ_2	$2\epsilon - 2\eta$	$2\epsilon - 2\delta$	$2\epsilon_1 + 2\delta_1$
$\epsilon_2\epsilon_3$	$4\epsilon^2 - 6\epsilon\eta + 2\xi^2$	2ξ	$4\epsilon_1^2 + 6\epsilon_1\delta_1 - 2\delta_1^2 + 2\delta_2$

Table 1: Conversion table for different definitions of the expansion basis.

[a] $\epsilon \to \epsilon_1$ and $\delta \to \delta_1$

To proceed with the calculation of the inflationary perturbations, we have to relate the comoving Hubble rate, aH, to conformal time. From the definition $\mathrm{d}\tau \equiv \mathrm{d}t/a$ we find after a partial integration

$$\tau = -\frac{1}{aH(1-\epsilon_1)} + \int \frac{\epsilon_1\epsilon_2}{(1-\epsilon_1)^2}\frac{\mathrm{d}N}{aH}. \tag{5}$$

The equation of motion of the perturbations is (see [14, 9] for the notation):

$$\mu(k,\tau)'' + (k^2 - \frac{z''}{z})\mu(k,\tau) = 0, \tag{6}$$

where a prime denotes a derivative with respect to conformal time and where $z = a\sqrt{\epsilon_1}$ for scalar perturbations and $z = a$ for tensor perturbations. This equation should be solved with the following initial conditions

$$\lim_{k/(aH)\to+\infty} \mu_{\mathrm{S,T}}(\tau) = \mp 4\sqrt{\pi}\, l_{\mathrm{Pl}} \frac{e^{-ik(\tau-\tau_{\mathrm{i}})}}{\sqrt{2k}}, \tag{7}$$

where l_{Pl} denotes the Planck length (the two signs stand for scalar and tensor perturbations respectively). Then the power spectra can be calculated and read

$$k^3 P_\zeta = \frac{k^3}{8\pi^2}\left|\frac{\mu_{\mathrm{S}}}{z_{\mathrm{S}}}\right|^2, \quad k^3 P_h = \frac{2k^3}{\pi^2}\left|\frac{\mu_{\mathrm{T}}}{z_{\mathrm{T}}}\right|^2, \tag{8}$$

where ζ and h stand for scalar and tensorial modes respectively.

3 Scalar perturbations

The potential in the scalar mode equation reads

$$\frac{z''}{z} = a^2H^2\left(2 - \epsilon_1 + \frac{3}{2}\epsilon_2 + \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_1\epsilon_2 + \frac{1}{2}\epsilon_2\epsilon_3\right). \tag{9}$$

We can solve the mode equations by Bessel functions

$$\mu = (k\tau)^{1/2}[B_1 J_\nu(k\tau) + B_2 J_{-\nu}(k\tau)]\,, \tag{10}$$

with the constants B_1 and B_2, if

$$\tau^2 \frac{z''}{z} \equiv \nu^2 - \frac{1}{4} \tag{11}$$

may be approximated to be constant. This condition is met if we can consistently neglect the time derivative of ϵ_1 and ϵ_2, which means that we have to set $\epsilon_1\epsilon_2 \approx 0$ and $\epsilon_2\epsilon_3 \approx 0$ from equations (4). Note that this does not imply $\epsilon_n \approx 0$. We see from equation (5) that we actually need to require that $\epsilon_1\epsilon_2\Delta N < A/100\%$, where A is the accuracy that we want to achieve and ΔN is the number of e-folds during which the time derivatives have to be neglected. Dropping the above mentioned terms in Eqs. (9) and (5) we find the index of the Bessel functions, which is given by

$$\nu \approx \frac{1}{2} + \frac{1}{1-\epsilon_1} + \frac{1}{2}\epsilon_2. \tag{12}$$

In the above expression, $1/(1-\epsilon_1)$ should be expanded for small ϵ_1 and truncated at order n, such that $\epsilon_1^n > |\epsilon_1\epsilon_2|$.

We can now fix B_1, B_2 by comparing to Eq. (7) in the limit $\tau \to \tau_{\rm i}$. This gives

$$B_1 = 2\pi l_{\rm Pl} \frac{\exp[i(\nu\pi/2 + \pi/4 + k\tau_{\rm i})]}{\sqrt{k}\sin(\pi\nu)}, \quad B_2 = -\exp(-i\pi\nu)B_1. \tag{13}$$

The next step is to calculate the superhorizon limit of μ, which becomes

$$|\mu| \to \frac{l_{\rm Pl}\pi 2^{\nu+1}(-k\tau)^{1/2-\nu}}{\sqrt{k}\sin(\pi\nu)\Gamma(1-\nu)} = \frac{l_{\rm Pl}2^{\nu+1}(-k\tau)^{1/2-\nu}\Gamma(\nu)}{\sqrt{k}}. \tag{14}$$

Inserting in the definition of the power spectrum (8) and eliminating τ with help of Eq. (5) we find

$$k^3 P_\zeta \approx \frac{l_{\rm Pl}^2 H^2}{\pi\epsilon_1}\left[\frac{1}{\pi}(2-2\epsilon_1)^{(2\nu-1)}\Gamma^2(\nu)\right]\left(\frac{k}{aH}\right)^{(3-2\nu)}. \tag{15}$$

At this point we have to fix the time t_* at which we evaluate ϵ_m. We introduce the abbreviation $k_* \equiv (aH)(t_*)$. We have shown above, that terms like $\epsilon_1\epsilon_2$

should be neglected consistently. However, the above result certainly still involves terms like this. In the last step we therefore have to expand Eq. (15) for ϵ_1 and ϵ_2 and drop all terms that are of order $\epsilon_1\epsilon_2$ or smaller. The crucial difference to the slow-roll approximation as used by Stewart and Lyth [7] is that they assume $\epsilon = \epsilon_1$ and $\delta = \epsilon_2/2 - \epsilon_1$ to be small simultaneously, thus they cannot keep higher power in their slow-roll parameters. In our approach we can, e.g., keep terms of order ϵ_m^5 if they are larger than $|\epsilon_1\epsilon_2|$ and $|\epsilon_2\epsilon_3|$. Consequently, depending on the relative magnitude of the functions ϵ_1, ϵ_2 and ϵ_3 we obtain different results.

3.1 Constant-horizon approximation

In some inflationary models as for inflation at a maximum, the time derivative of the Hubble distance is tiny. For this kind of models, during a certain number of e-folds, $\epsilon_1 \ll 1$. However, it does not necessarily mean that all other ϵ_m have to be small as well. We thus define the constant-horizon approximation at order n for the situation $|\epsilon_2^n| > \max(|\epsilon_1\epsilon_2|, |\epsilon_2\epsilon_3|)$, which means that we are allowed to include the following monomials in the primordial spectra: $1, \epsilon_1, \epsilon_2, \ldots, \epsilon_2^n$.

Expanding Eq. (15) we find at third order ($n = 3$)

$$k^3 P_\zeta \approx \frac{l_{\rm Pl}^2 H^2}{\pi\epsilon_1}\left[a_0 + a_1 \ln\left(\frac{k}{k_*}\right) + a_2 \ln^2\left(\frac{k}{k_*}\right) + a_3 \ln^3\left(\frac{k}{k_*}\right) + \ldots\right] \quad (16)$$

$$a_0 = 1 - 2(C+1)\epsilon_1 - C\epsilon_2 + \frac{1}{8}(4C^2 + \pi^2 - 8)\epsilon_2^2 - \frac{1}{24}\left[4C^3 - 3C(8-\pi^2) + 14\zeta(3) - 16\right]\epsilon_2^3\,, \quad (17)$$

$$a_1 = -2\epsilon_1 - \epsilon_2 + C\epsilon_2^2 - \frac{1}{8}(4C^2 + \pi^2 - 8)\epsilon_2^3\,, \quad (18)$$

$$a_2 = \frac{1}{2}\epsilon_2^2 - \frac{1}{2}C\epsilon_2^3\,, \quad (19)$$

$$a_3 = -\frac{1}{6}\epsilon_2^3\,, \quad (20)$$

where $C \equiv \gamma_{\rm E} + \ln 2 - 2 \approx -0.7296$ and $\zeta(3) \approx 1.2021$. The quantity that is usually called the amplitude is obtained by setting $k = k_*$. We finally calculate the spectral index

$$n_{\rm S} - 1 \equiv \frac{\mathrm{d}\ln k^3 P_\zeta}{\mathrm{d}\ln k}|_{k=k_*}\,, \quad (21)$$

which is most easily obtained from Eq. (15). It reads $n_S - 1 = -2\epsilon_1 - \epsilon_2$ at any order n in the constant horizon approximation. In the corresponding limits this result agrees with the usual slow-roll expression. Consistently with our assumptions, there is no "running" of the spectral index.

3.2 Growing-horizon approximation

Power-law inflation ($a \propto t^p$) is one of the few models of inflation for which an exact expression for the primordial spectrum can be obtained in closed form. Our horizon flow functions in this case are $\epsilon_1 = 1/p = const.$ and $\epsilon_m = 0$ for $m > 1$. This observation suggests that inflationary models with $|\epsilon_m| < \epsilon_1$ for $m > 1$ might be approximated by keeping all terms in ϵ_1 up to order n, where n is the maximal integer for which $\epsilon_1^n > \max(|\epsilon_1\epsilon_2|, |\epsilon_2\epsilon_3|)$ holds true. For this kind of models, $\epsilon_1 \simeq const.$, implies that the Hubble radius grows almost like a linear function of time. Hence, we define the (linearly) growing-horizon approximation to order n to include the following terms: $1, \epsilon_1, \ldots, \epsilon_1^n, \epsilon_2$.

The scalar power spectrum at third order in the growing horizon approximation reads:

$$
\begin{aligned}
k^3 P_\zeta &\approx \frac{l_{\rm Pl}^2 H^2}{\pi\epsilon_1}\left[b_0 + b_1 \ln\left(\frac{k}{k_*}\right) + b_2 \ln^2\left(\frac{k}{k_*}\right) + b_3 \ln^3\left(\frac{k}{k_*}\right) + \ldots\right] && (22)\\
b_0 &= 1 - 2(C+1)\epsilon_1 + \frac{1}{2}\left[4C(C+1) + \pi^2 - 10\right]\epsilon_1^2 \\
&\quad - \frac{1}{3}\left[4C^3 + 3C(\pi^2 - 12) + 14\zeta(3) - 19\right]\epsilon_1^3 - C\epsilon_2\,, && (23)\\
b_1 &= -2\epsilon_1 + 2(2C+1)\epsilon_1^2 - (4C^2 + \pi^2 - 12)\epsilon_1^3 - \epsilon_2\,, && (24)\\
b_2 &= 2\epsilon_1^2 - 4C\epsilon_1^3\,, && (25)\\
b_3 &= -\frac{4}{3}\epsilon_1^3. && (26)
\end{aligned}
$$

Ignoring the monomials ϵ_1^2 and ϵ_1^3, but keeping ϵ_2, we recover the result of Stewart and Lyth [7]. The spectral index at n-th order reads

$$n_S - 1 = 3 - 2\nu \approx -2(\epsilon_1 + \epsilon_1^2 + \epsilon_1^3 + \ldots + \epsilon_1^n) - \epsilon_2, \qquad (27)$$

and again there is no "running" of the spectral index.

4 Tensor perturbations

Now the potential in the mode equation reads

$$\frac{z''}{z} = (aH)^2 \left(2 - \epsilon_1\right), \tag{28}$$

which allows us to solve the mode equation with the same approximation as in the scalar case, but with Bessel function index

$$\nu \approx \frac{1}{2} + \frac{1}{1-\epsilon_1} \ . \tag{29}$$

The following steps are analogous to those for the calculation of the scalar perturbations, the basic difference is that no contribution of ϵ_2 arises in any case.

Using the constant-horizon approximation we obtain

$$\begin{aligned} k^3 P_h &= \frac{16 l_{\mathrm{Pl}}^2 H^2}{\pi}\left[\mathtt{a}_0 + \mathtt{a}_1 \ln\left(\frac{k}{k_*}\right) + \ldots\right], & (30)\\ \mathtt{a}_0 &= 1 - 2(C+1)\epsilon_1, & (31)\\ \mathtt{a}_1 &= -2\epsilon_1, & (32)\\ n_{\mathrm{T}} &= -2\epsilon_1, & (33) \end{aligned}$$

at any order. Note that all $\mathtt{a}_i$ for $i \geq 2$ vanish in the constant horizon approximation and that there are no higher order corrections to n_{T}, which follows from the absence of any ϵ_2-dependence in the potential (28) and the fact that higher powers of ϵ_1 are consistently neglected. The growing-horizon approximation leads to

$$\begin{aligned} k^3 P_h &= \frac{16 l_{\mathrm{Pl}}^2 H^2}{\pi}\left[\mathtt{b}_0 + \mathtt{b}_1 \ln\left(\frac{k}{k_*}\right) + \mathtt{b}_2 \ln^2\left(\frac{k}{k_*}\right) + \mathtt{b}_3 \ln^3\left(\frac{k}{k_*}\right) \ldots\right] & (34)\\ \mathtt{b}_0 &= 1 - 2(C+1)\epsilon_1 + \frac{1}{2}\left[4C(C+1) + \pi^2 - 10\right]\epsilon_1^2 & \\ &\quad - \frac{1}{3}\left[4C^3 + 3C(\pi^2 - 12) + 14\zeta(3) - 19\right]\epsilon_1^3, & (35)\\ \mathtt{b}_1 &= -2\epsilon_1 + 2(2C+1)\epsilon_1^2 - (4C^2 + \pi^2 - 12)\epsilon_1^3, & (36)\\ \mathtt{b}_2 &= 2\epsilon_1^2 - 4C\epsilon_1^3, & (37)\\ \mathtt{b}_3 &= -\frac{4}{3}\epsilon_1^3, & (38) \end{aligned}$$

at third order, and

$$n_{\mathrm{T}} = -2(\epsilon_1 + \epsilon_1^2 + \epsilon_1^3 + \cdots + \epsilon_1^n), \tag{39}$$

at nth order. Again both results are consistent with the results of Stewart and Lyth [7].

5 Consistency relations

It is interesting to inspect the so-called 'consistency relations of inflation' for models that do not belong to the class of slow-roll inflation. We define the tensor-to-scalar ratio $r \equiv P_h/P_\zeta$. The 'classic' result for slow-roll models reads at first order $n_{\mathrm{T}} = -r/8$. We find for the constant horizon and for the growing horizon approximations $r = 16\epsilon_1$ at any order. This result may be used to express the corresponding expressions for the tensorial spectral index as

$$n_{\mathrm{T}} = -\frac{r}{8} \tag{40}$$

for the constant horizon approximation at any order, and

$$n_{\mathrm{T}} = -2\left[\left(\frac{r}{16}\right) + \left(\frac{r}{16}\right)^2 + \ldots + \left(\frac{r}{16}\right)^n\right] \tag{41}$$

for the growing horizon approximation at order n. Both results are consistent with the second order result of Ref. [12], $n_{\mathrm{T}} = -2[(r/16) - (r/16)^2 - (n_{\mathrm{S}} - 1)(r/16)]$, since $(n_{\mathrm{S}} - 1)(r/16) = -2(r/16)^2 + \mathcal{O}(\epsilon_1\epsilon_2)$. For models where the constant horizon approximation applies we expect that $r \ll 1$, thus the prospects to detect the tensors and therefore to test the consistency relation are bad. In the case of those models for which the growing horizon approximation is suited, there is a chance to detect the tensor contribution. Here, the corrections to the slow-roll results given by Eq. (41) may be relevant. However, current data seem to indicate that $r < 1$, which implies that higher order corrections to the classic consistency relation are not important.

6 Conclusion

We may compare our result with the work by Stewart and Gong [11]. They used a new method to obtain the scalar amplitude at second order in the

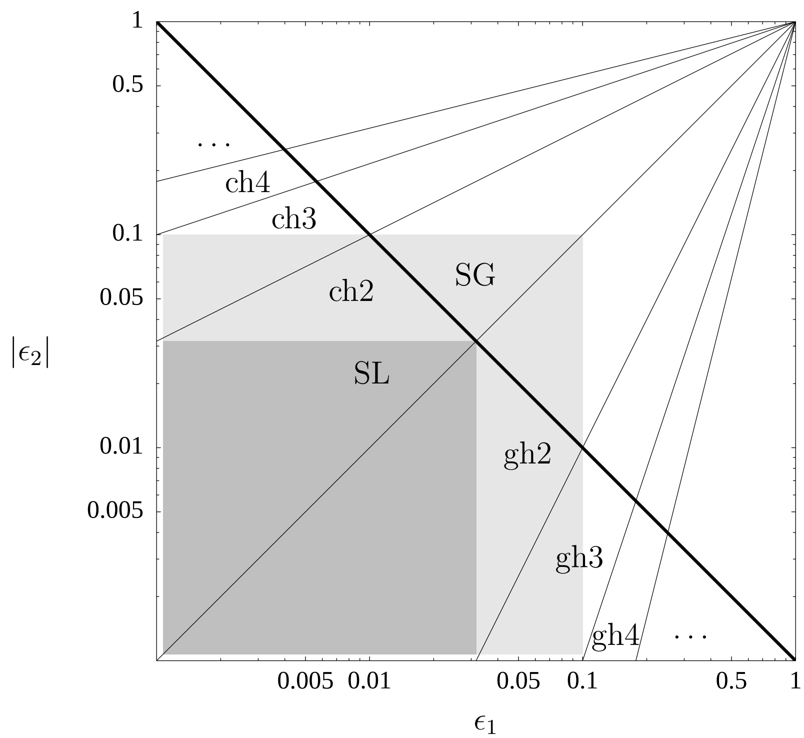

Figure 1: Regions in the ϵ_1-$|\epsilon_2|$ parameter space where the spectral amplitudes can be calculated with an accuracy better than 1%. In the dark shaded region the Stewart-Lyth (SL) approximation [7], as well as all other approximations are fine. Second-order corrections, as calculated by Stewart and Gong (SG) [11], extend that region to the light shaded region. The constant horizon approximation at order n (ch$n$), and the growing horizon approximation at order n (gh$n$), do well below the thick line. The rays indicate where the corresponding higher order corrections are necessary. The thick line itself is the condition $\epsilon_1|\epsilon_2| < (A/100\%)/\Delta N$, with $\Delta N = 10$ and $A = 1\%$. As is easily seen the ch2 and gh2 regions are included within the SG region. The ch$n$ and gh$n$ regions with $n > 2$ allow us to go beyond the SG approximation.

slow-roll parameters. In our notation they obtain:

$$
\begin{aligned}
k^3 P_\zeta|_{k=k_*} \approx & \frac{l_{\rm Pl}^2 H^2}{\pi\epsilon_1}\left\{1 - 2(C+1)\epsilon_1 - C\epsilon_2\right. \\
& + \left[2C(C+1) + \frac{\pi^2}{2} - 5\right]\epsilon_1^2 + \left(\frac{1}{2}C^2 + \frac{\pi^2}{8} - 1\right)\epsilon_2^2 \\
& \left. + \left[C(C-1) + \frac{7\pi^2}{12} - 7\right]\epsilon_1\epsilon_2 + \left(-\frac{1}{2}C^2 + \frac{\pi^2}{24}\right)\epsilon_2\epsilon_3\right\}, \quad (42)
\end{aligned}
$$

which agrees with our result if $\epsilon_1\epsilon_2 \approx 0$ and $\epsilon_2\epsilon_3 \approx 0$.

Our results can be applied to many interesting models of inflation. The constant horizon approximation works fine if the inflaton field sits close to a maximum of the potential. On the other hand the growing horizon approximation is a good approximation for chaotic inflation models with monomial potential $\propto \phi^p$, when $p > 4$. To give an example, for $p = 8$ we have $\epsilon_1 \approx 0.04$ and $\epsilon_2 \approx 0.02$, which allows us to take all quadratic terms in ϵ_1 into account. We plot in figure 1 the domain of applicability of the various approximations in the ϵ_1-$|\epsilon_2|$ plane. We tacitly assume that $|\epsilon_3| < \min[\epsilon_1, |\epsilon_2|]$, for the sake of simplicity of the argument. If this condition is not satisfied an analogous three dimensional plot has to replace figure 1.

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